# Exact WKB Analysis of Discrete Quantum Systems and Its Applications to Optimal Control

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**TS**, Taniguchi, Iwamura, PRA (2024)
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# Adiabatic Quantum System

**Schrödinger equation**

$$i\frac{d}{dt}|\psi(t)\rangle = H\left(\frac{t}{T}\right)|\psi(t)\rangle$$

$T \to \infty$ : Adiabatic limit

- Quantum Control

Intelligent Computing
A SCIENCE PARTNER JOURNAL

RESEARCH ARTICLE

**Quantification of Robustness, Leakage, and Seepage for Composite and Adiabatic Gates on Modern NISQ Systems**

Kajsa Williams[1,2*] and Louis-S. Bouchard[1,2,3*] (2024)

Adiabatic approximation :

$$|\psi(t_0)\rangle = |E_n(t_0)\rangle \Rightarrow |\psi(t)\rangle \simeq e^{-iT\int_{t_0/T}^{t/T} ds E_n(s)} |E_n(t)\rangle$$

Dynamical phase

$$\left( H\left(\frac{t}{T}\right)|E_n(t)\rangle = E_n\left(\frac{t}{T}\right)|E_n(t)\rangle \right)$$

# Singular Perturbation Theory

**Schrödinger equation**

$$i\frac{d}{dt}|\psi(t)\rangle = H\left(\frac{t}{T}\right)|\psi(t)\rangle$$

$$\tau := \frac{t}{T}$$

$$\frac{i}{T}\frac{d}{d\tau}|\psi(\tau)\rangle = H(\tau)|\psi(\tau)\rangle$$

As $T \to \infty$, order of differential equation changes → Singular perturbation theory

For example: One-dimensional stationary Schrödinger equation （$\hbar \to 0$ : WKB approximation）

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)|\psi(x)\rangle = E|\psi(x)\rangle$$

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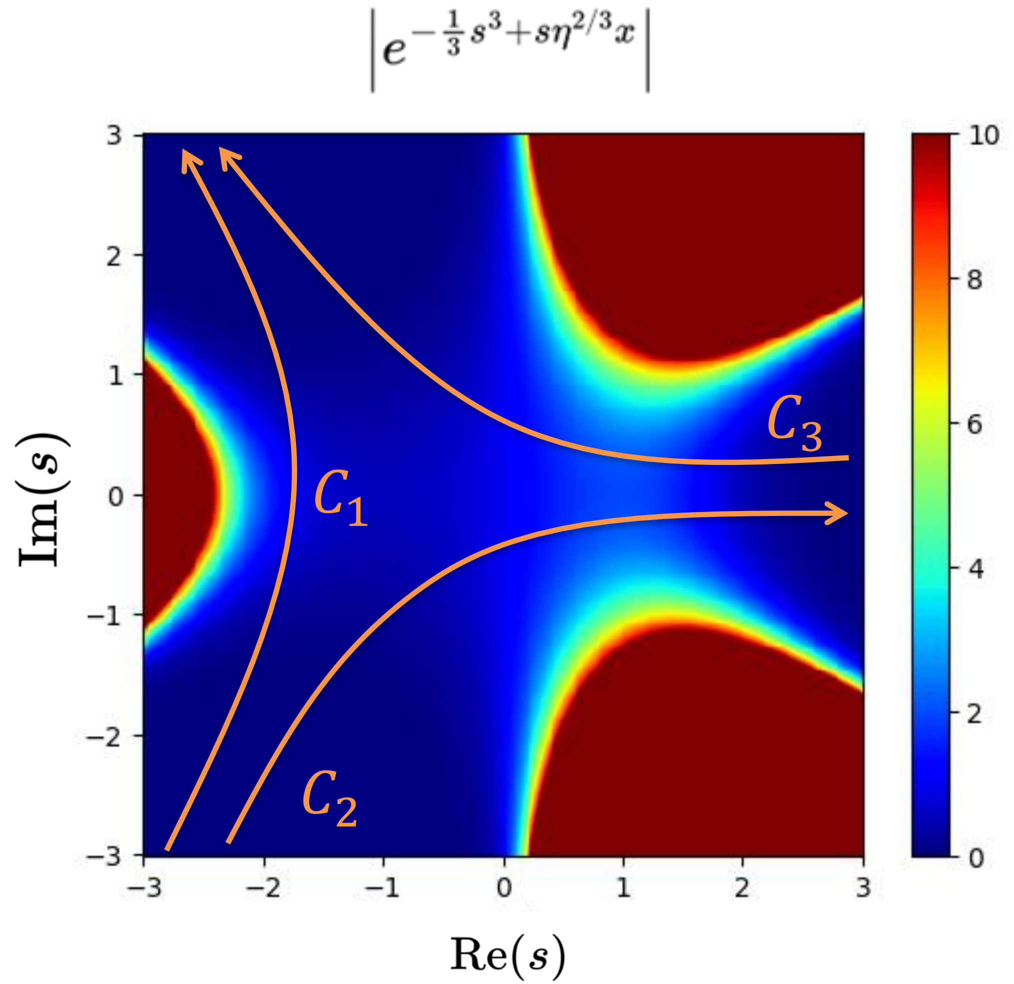
# Airy Function

**Airy Equation:** $$\left(-\frac{\partial^2}{\partial x^2}+\eta^2 x\right)\psi(x,\eta)=0,\quad \eta>0$$

**Solutions:** $$\psi(x,\eta)=A\int_{C_i} e^{-\frac{1}{3}s^3+s\eta^{2/3}x}ds$$

Example: $$\mathrm{Ai}(\eta^{2/3}x)=\frac{1}{2\pi i}\int_{C_1} e^{-\frac{1}{3}s^3+s\eta^{2/3}x}ds$$




**WKB solutions：**

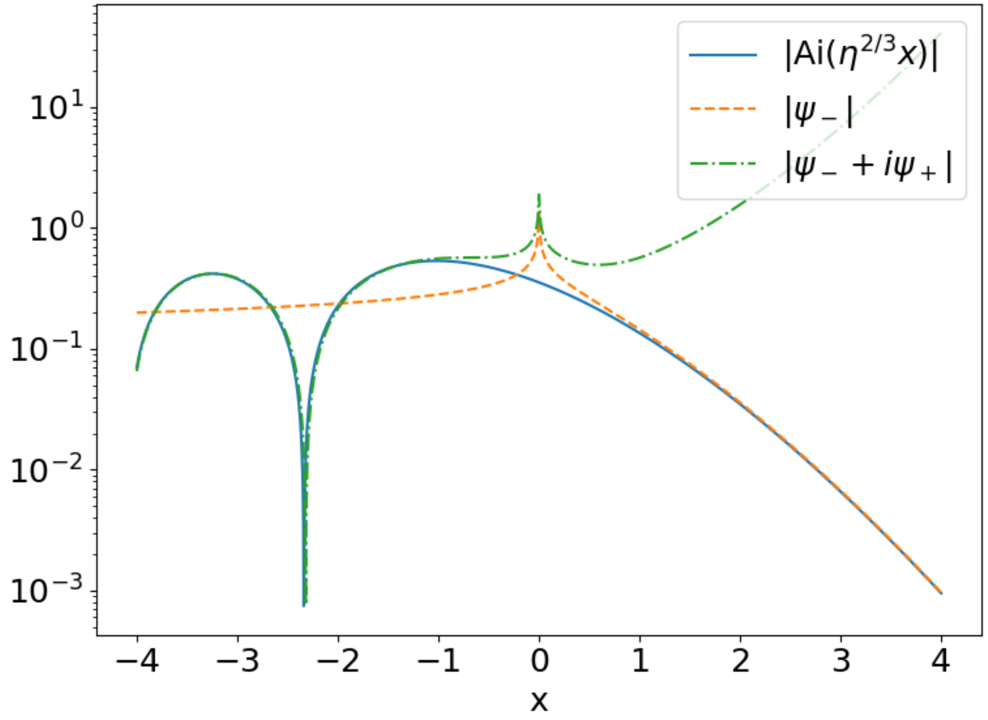
$$\psi_\pm(x,\eta):=x^{-\frac{1}{4}}e^{\pm\frac{2}{3}\eta x^{\frac{3}{2}}}$$

- Up to $O(\eta^0)$ in the asymptotic expansion
- Irregular singularity at $x=\infty$

# Asymptotic Behavior of the Airy Function

$$\psi_\pm(x,\eta) := x^{-\frac{1}{4}} e^{\pm\frac{2}{3}\eta x^{\frac{3}{2}}}$$

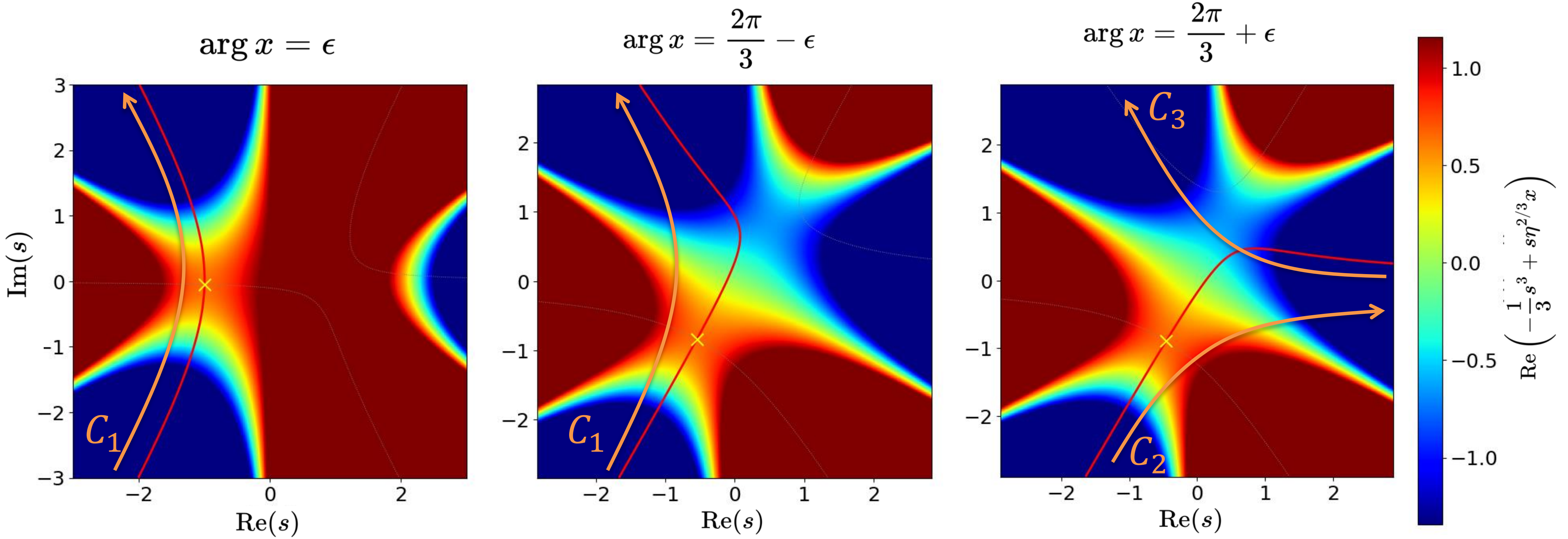
$$\mathrm{Ai}\left(\eta^{2/3}x\right) \sim \begin{cases} \frac{1}{2\sqrt{\pi}\eta^{1/6}}\psi_-(x,\eta) & x\to\infty \\ \frac{1}{2\sqrt{\pi}\eta^{1/6}}\left(\psi_-(x,\eta) + \boxed{i\psi_+(x,\eta)}\right) & x\to-\infty \end{cases}$$




When does this discontinuous change occur when $x$ is a complex variable?

# Stokes Phenomenon

$$\mathrm{Ai}(\eta^{2/3}x) = \frac{1}{2\pi i}\int_{C_1} e^{-\frac{1}{3}s^3 + s\eta^{2/3}x} ds \sim \begin{cases} \frac{1}{2\sqrt{\pi}\eta^{1/6}}\psi_-(x,\eta) & x \to \infty \\ \frac{1}{2\sqrt{\pi}\eta^{1/6}}(\psi_-(x,\eta) + i\psi_+(x,\eta)) & x \to -\infty \end{cases}$$



# Stokes Phenomenon

$$\psi_\pm(x,\eta) := x^{-\frac{1}{4}} e^{\pm\frac{2}{3}\eta x^{\frac{3}{2}}}$$

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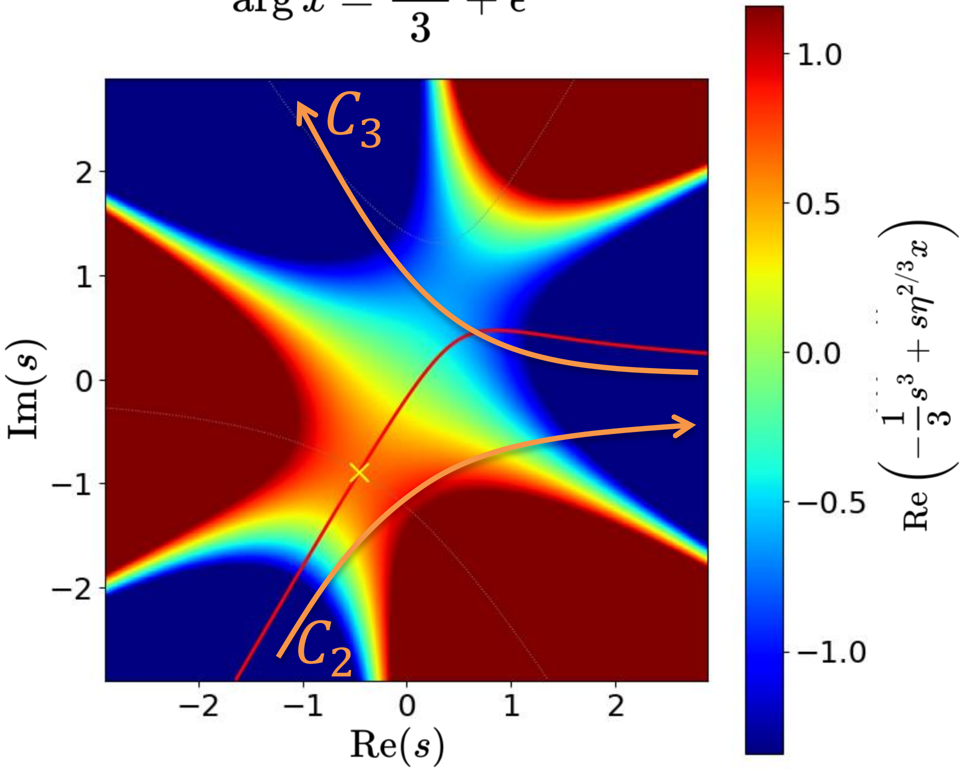
$\arg x = \frac{2\pi}{3} + \epsilon$




$C_2$

Exponentially large
due to the contribution from the saddle point

$C_3$

Exponentially small
because the contour avoids the saddle point

At $\arg x = \frac{2\pi}{3}$, $\psi_-(x,\eta)$ becomes dominant
→ an exponentially small, discontinuous change occurs

**Stokes phenomenon**

# Stokes Diagram for Airy Function

$$\psi_\pm(x,\eta) := x^{-\frac{1}{4}} e^{\pm\frac{2}{3}\eta x^{\frac{3}{2}}}$$

**Stokes phenomenon**:

$$\begin{aligned}
\psi_-(x,\eta) &\to \psi_-(x,\eta) + i\psi_+(x,\eta) \quad (\mathrm{III} \to \mathrm{I}) \\
\psi_+(x,\eta) &\to \psi_+(x,\eta) + i\psi_-(x,\eta) \quad (\mathrm{I} \to \mathrm{II}) \\
\psi_-(x,\eta) &\to \psi_-(x,\eta) + i\psi_+(x,\eta) \quad (\mathrm{II} \to \mathrm{III})
\end{aligned}$$

Counterclockwise:

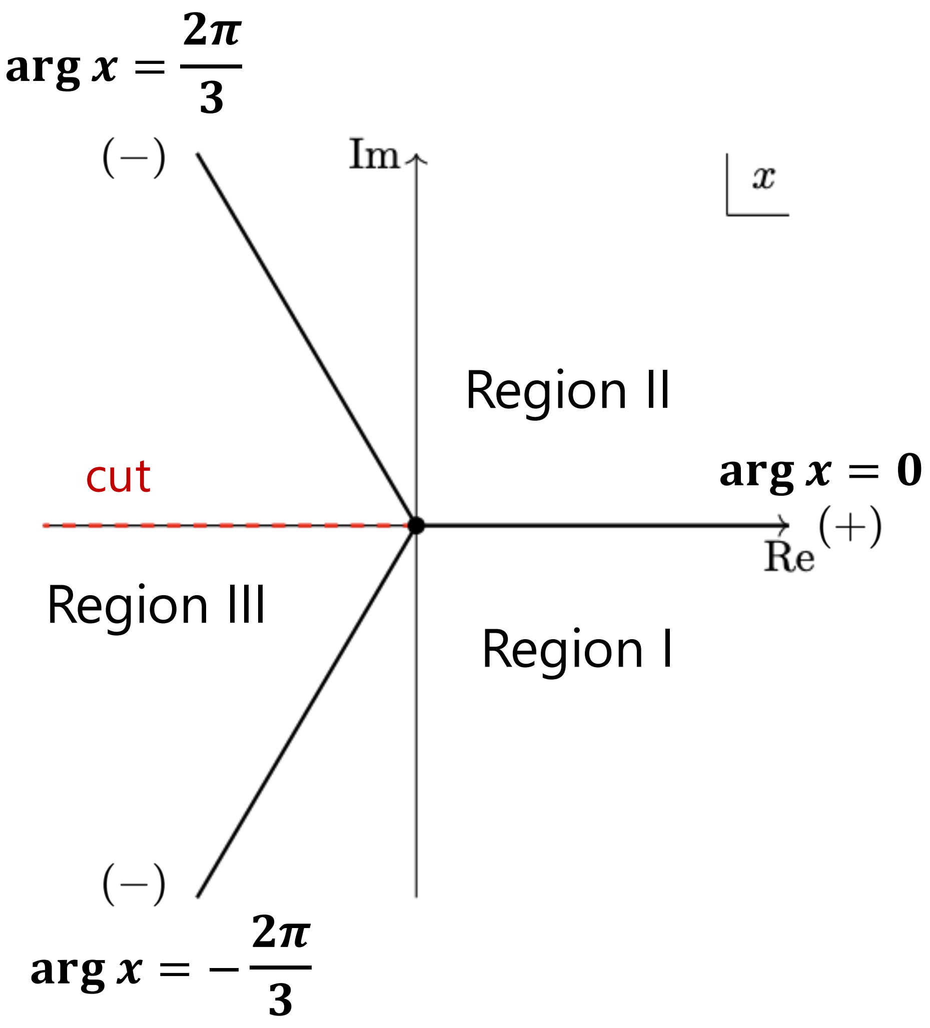
$$(\text{ dominant }) \to (\text{ dominant }) + i \times (\text{ subdominant })$$

**Stokes lines**:

Region where the Stokes phenomenon occurs

**Turning points**:

The points from which Stokes lines emerge



# Key Results of Exact WKB Analysis

$$\left(-\frac{\partial^2}{\partial x^2}+\eta^2 Q(x)\right)\psi(x,\eta)=0,\quad \eta>0$$

**Turning points**: Points $x_c$ where $Q(x_c)=0$

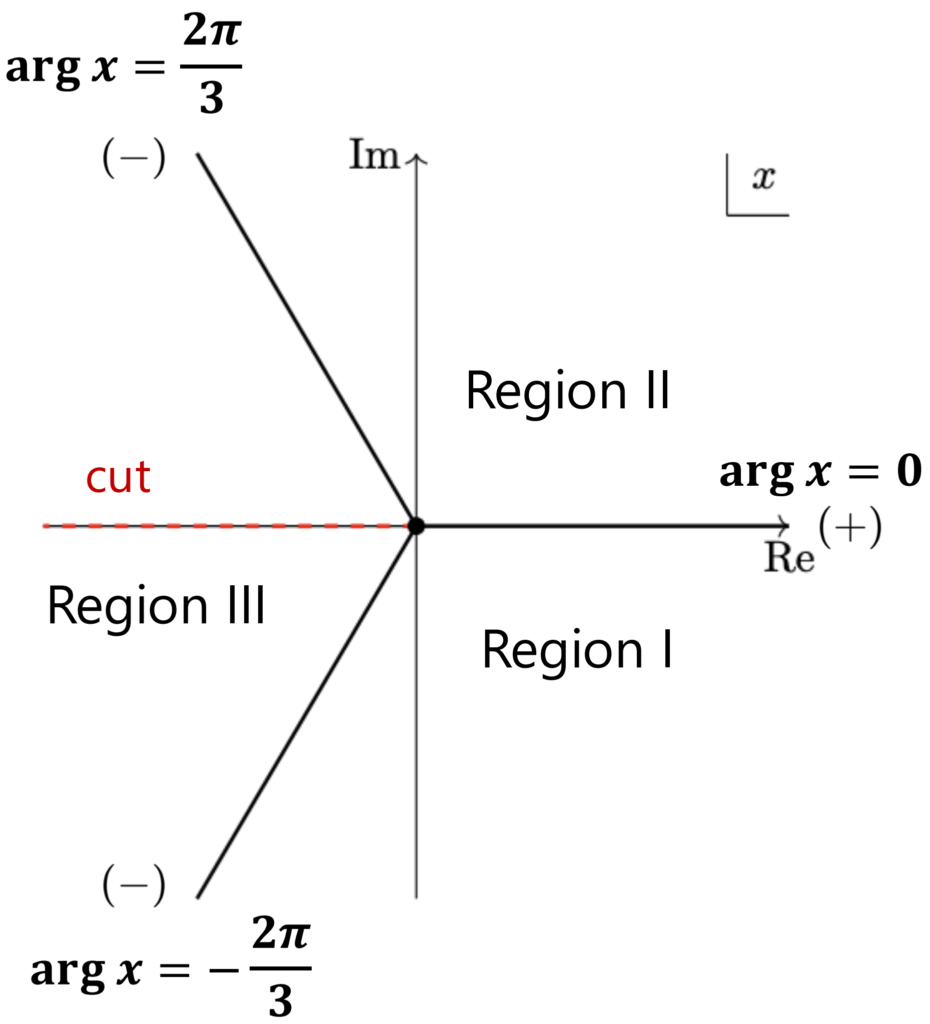
**Stokes lines**:

The set of points $x$ satisfying $\mathrm{Im}\int_{x_c}^{x}\sqrt{Q(s)}ds=0$

※ WKB solutions: $\psi_\pm(x,x_c,\eta)\sim\exp\left(\pm\int_{x_c}^{x}\eta\sqrt{Q(s)}ds\right)$

For Airy equation, turning points: $x_c=0$

Stokes lines: $\mathrm{Im}\int_0^x s^{1/2}ds=\frac{2}{3}\mathrm{Im}\,x^{3/2}=0$



# Key Results of Exact WKB Analysis

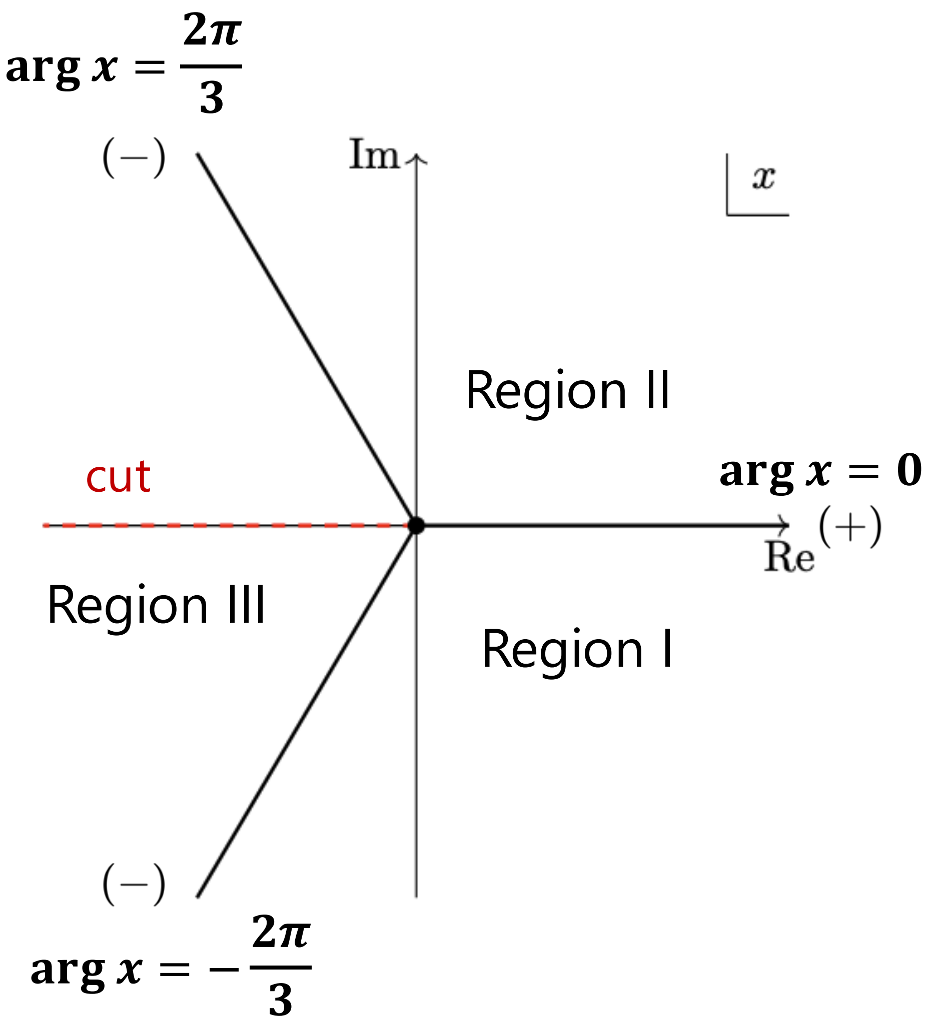
$$\left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q(x)\right)\psi(x,\eta) = 0, \quad \eta > 0$$

**Turning points**: Points $x_c$ where $Q(x_c) = 0$

**Stokes lines**:

The set of points $x$ satisfying $\mathrm{Im} \int_{x_c}^{x} \sqrt{Q(s)}ds = 0$

**Theorem**:
If three Stokes lines emerge from a turning point
and each extends to infinity without intersecting any
other turning point,
then the connection formula is the same as that of the
Airy function.



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# Analysis of the Nonlinear LZSM Model in the Adiabatic Limit

**(Nonlinear) Landau-Zener-Stückelberg-Majorana model:**

$$i\frac{\partial}{\partial t}|\psi(t,\eta)\rangle = \eta H(t)|\psi(t,\eta)\rangle, \quad t \in [t_I, t_F]$$

$$H(t) = t^n \sigma_z + \kappa \sigma_x, \quad (n = 1, 3, 5, \cdots)$$

Adiabatic limit : $\eta \to \infty$

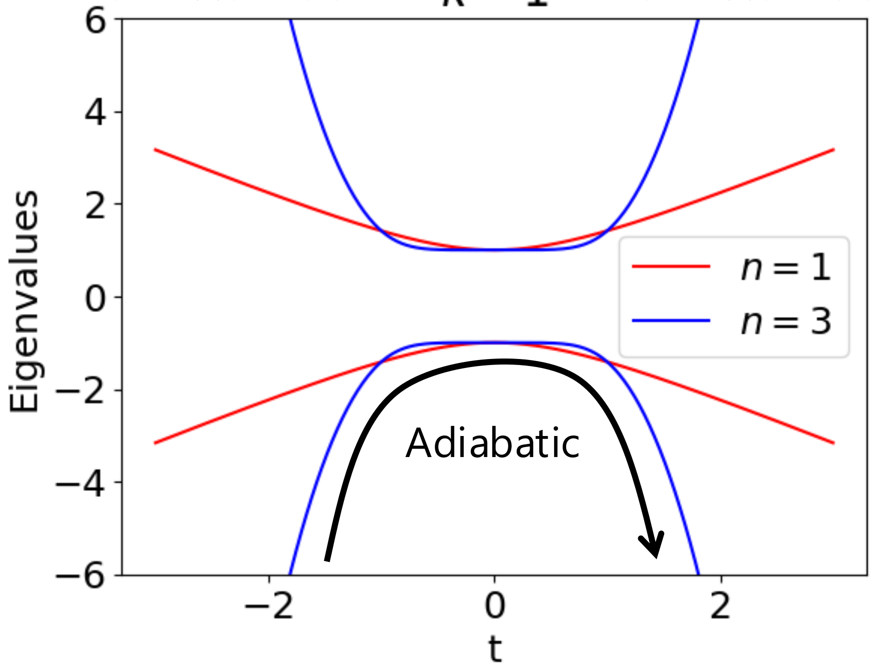
$$\left(\because \quad i\frac{\partial}{\partial \tau}|\psi(\tau,\eta)\rangle = H\left(\frac{\tau}{\eta}\right)|\psi(\tau,\eta)\rangle, \quad \tau = \eta t\right)$$

**Goal: To approximate the time-evolution operator**

Note: In the adiabatic approximation,

$$\langle E_+(t_F)|U(t_F, t_I)|E_-(t_I)\rangle \simeq 0$$

$|E_+(t)\rangle = |2\rangle$ $\kappa = 1$ $|E_+(t)\rangle = |1\rangle$




$(\ \sigma_z|1\rangle = |1\rangle, \ \ \sigma_z|2\rangle = -|2\rangle\ )$

# WKB Solutions for the Nonlinear LZSM Model

$$i\frac{\partial}{\partial t}|\psi(t,\eta)\rangle = \eta H(t)|\psi(t,\eta)\rangle, \quad H(t) = t^n\sigma_z + \kappa\sigma_x, \quad (n = 1,3,5,\cdots)$$

$$|\psi(t,\eta)\rangle = \begin{pmatrix} a(t,\eta) \\ b(t,\eta) \end{pmatrix}$$

$$\left(\frac{\partial^2}{\partial t^2} + \eta^2\left(E^2(t) + \frac{1}{\eta}int^{n-1}\right)\right)a(t,\eta) = 0, \quad E(t) = \sqrt{t^{2n}+\kappa^2}$$

WKB solutions : $a_j(t,t_0,\eta) = \frac{1}{\sqrt{E(t)}}\exp\left((-1)^{j+1}\int_{t_0}^{t}\left(i\eta E(s) - \frac{ns^{n-1}}{2E(s)}\right)ds\right) \quad (j=1,2)$

$$|\psi_j(t,t_0,\eta)\rangle = \begin{pmatrix} 1 \\ \frac{1}{\kappa}\left((-1)^j E(t) - t^n\right) \end{pmatrix}\frac{1}{\sqrt{E(t)}}\exp\left((-1)^{j+1}\int_{t_0}^{t}\left(i\eta E(s) - \frac{ns^{n-1}}{2E(s)}\right)ds\right) \quad (j=1,2)$$

# Stokes Diagram for the Nonlinear LZSM Model

WKB solutions：

$$|\psi_j(t,t_0,\eta)\rangle = \begin{pmatrix} 1 \\ \frac{1}{\kappa}\left((-1)^j E(t) - t^n\right) \end{pmatrix} \frac{1}{\sqrt{E(t)}} \exp\left((-1)^{j+1}\int_{t_0}^{t}\left(i\eta E(s) - \frac{n s^{n-1}}{2E(s)}\right)ds\right)$$

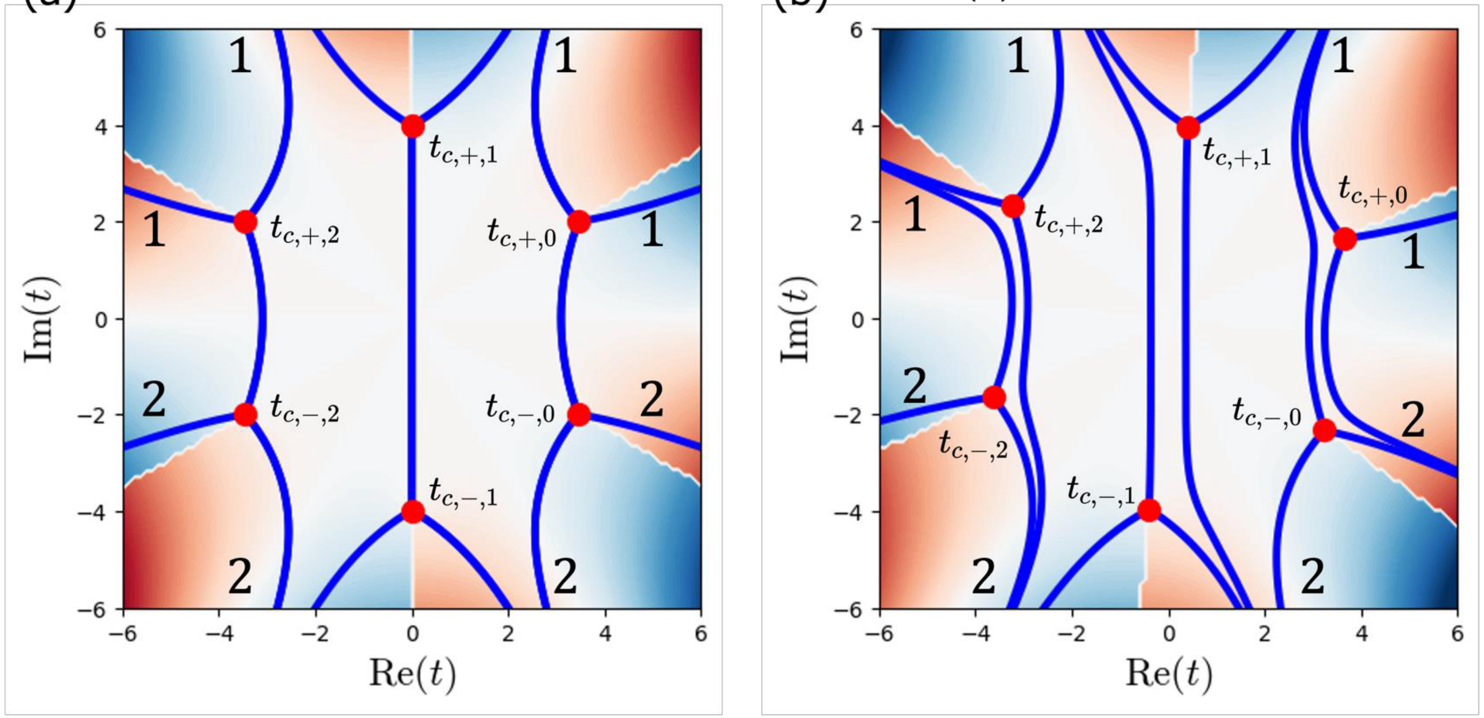
**Turning points**:

$$E(t_c) = 0 \Rightarrow t_{c,\pm,m} = e^{\pm i\left(\frac{\pi}{2n}+\frac{m}{n}\pi\right)}\kappa^{\frac{1}{n}}$$
$$(m = 0, 1, \cdots, n-1)$$

**Stokes lines**:

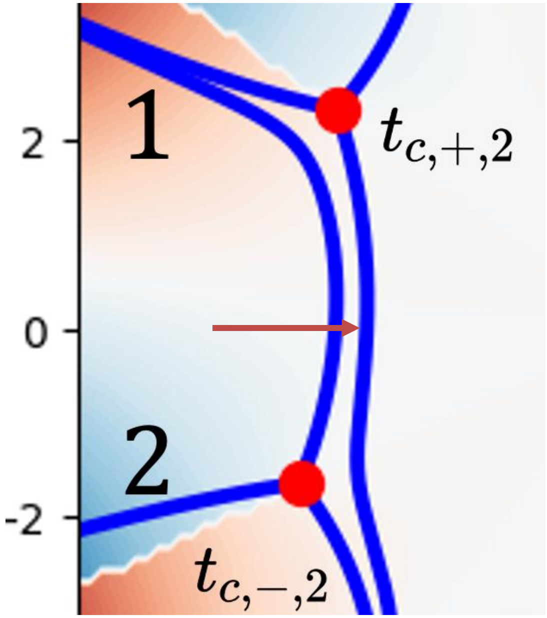
$$\mathrm{Re}\int_{t_{c,\pm,m}}^{t} E(s)ds = 0$$

(a) $H(t) = t^n\sigma_z + \kappa\sigma_x$ (b) $H(t) = e^{i\epsilon}t^n\sigma_z + \kappa\sigma_x$




Solution → Fujimori, et al. arXiv:2504.12838

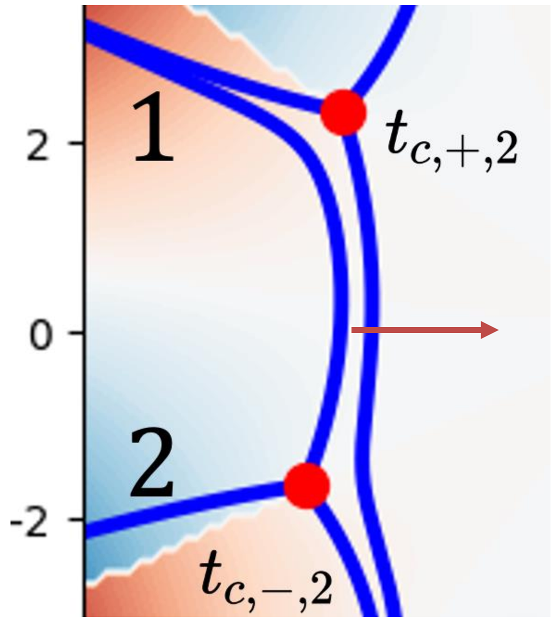
# Connection Formula for the Nonlinear LZSM Model




$$|\psi_1(t, t_{c,-,2}, \eta)\rangle \to |\psi_1(t, t_{c,-,2}, \eta)\rangle - i\,|\psi_2(t, t_{c,-,2}, \eta)\rangle, \quad |\psi_2(t, t_{c,-,2}, \eta)\rangle \to |\psi_2(t, t_{c,-,2}, \eta)\rangle$$

$$\Leftrightarrow \quad \begin{pmatrix} |\psi_1(t, t_{c,-2}, \eta)\rangle \\ |\psi_2(t, t_{c,-2}, \eta)\rangle \end{pmatrix} \to \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |\psi_1(t, t_{c,-2}, \eta)\rangle \\ |\psi_2(t, t_{c,-2}, \eta)\rangle \end{pmatrix}$$

# Connection Formula for the Nonlinear LZSM Model




$$|\psi_2(t, t_{c,+,2}, \eta)\rangle \to |\psi_2(t, t_{c,+,2}, \eta)\rangle + i\,|\psi_1(t, t_{c,+,2}, \eta)\rangle, \qquad |\psi_1(t, t_{c,+,2}, \eta)\rangle \to |\psi_1(t, t_{c,+,2}, \eta)\rangle$$

$$\Leftrightarrow \quad \begin{pmatrix} |\psi_1(t, t_{c,+,2}, \eta)\rangle \\ |\psi_2(t, t_{c,+,2}, \eta)\rangle \end{pmatrix} \to \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \begin{pmatrix} |\psi_1(t, t_{c,+,2}, \eta)\rangle \\ |\psi_2(t, t_{c,+,2}, \eta)\rangle \end{pmatrix}$$

# Connection Formula for the Nonlinear LZSM Model

$$\begin{pmatrix} |\psi_1(t,a,\eta)\rangle \\ |\psi_2(t,a,\eta)\rangle \end{pmatrix} = \underline{\begin{pmatrix} R_1(a,b)Q_1(a,b) & 0 \\ 0 & R_2(a,b)Q_2(a,b) \end{pmatrix}} \begin{pmatrix} |\psi_1(t,b,\eta)\rangle \\ |\psi_2(t,b,\eta)\rangle \end{pmatrix}$$

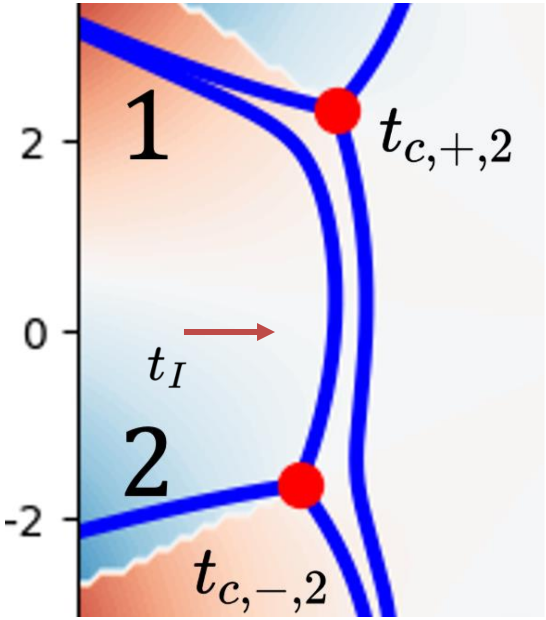
$$= \mathcal{N}(a,b)$$

$$R_j(a,b) = \exp\left((-1)^{j+1}\int_a^b i\eta E(s)ds\right), \quad Q_j(a,b) = \exp\left((-1)^j \int_a^b \frac{ns^{n-1}}{2E(s)}ds\right)$$

$$|\psi_j(t,t_0,\eta)\rangle = \left(\frac{1}{\frac{1}{\kappa}\left((-1)^j E(t) - t^n\right)}\right)\frac{1}{\sqrt{E(t)}}\exp\left((-1)^{j+1}\int_{t_0}^t \left(i\eta E(s) - \frac{ns^{n-1}}{2E(s)}\right)ds\right)$$

# Connection Formula for the Nonlinear LZSM Model

$$\begin{pmatrix} |\psi_1(t, t_I, \eta)\rangle \\ |\psi_2(t, t_I, \eta)\rangle \end{pmatrix} = \mathcal{N}(t_I, t_{c,-,2}) \begin{pmatrix} |\psi_1(t, t_{c,-,2}, \eta)\rangle \\ |\psi_2(t, t_{c,-,2}, \eta)\rangle \end{pmatrix}$$



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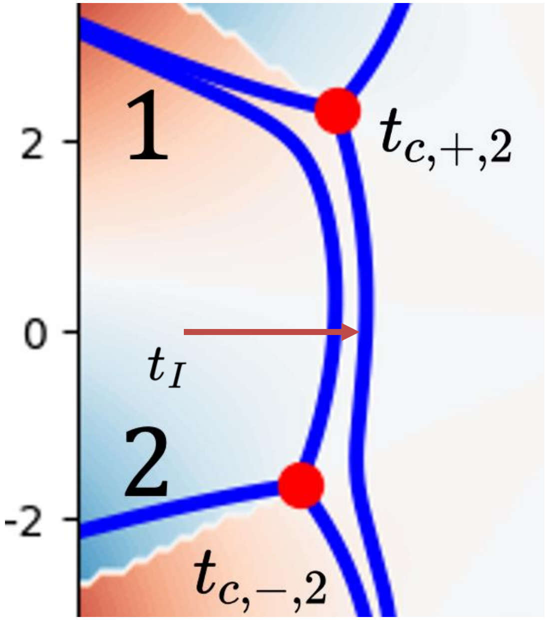
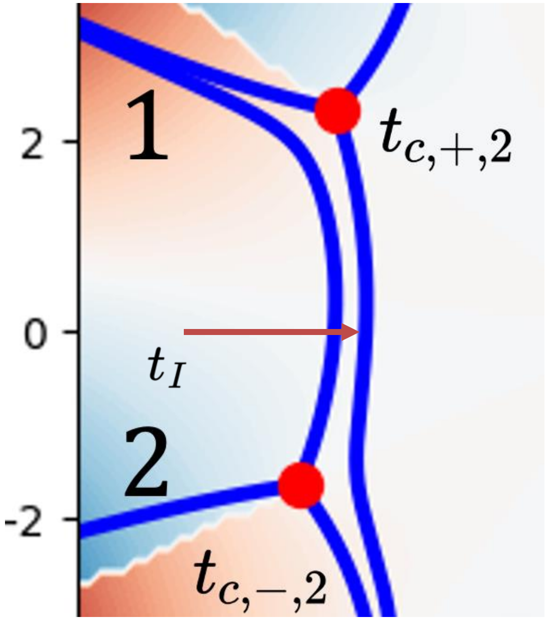
$$\rightarrow \mathcal{N}(t_I, t_{c,-,2}) \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |\psi_1(t, t_{c,-,2}, \eta)\rangle \\ |\psi_2(t, t_{c,-,2}, \eta)\rangle \end{pmatrix}$$



# Connection Formula for the Nonlinear LZSM Model

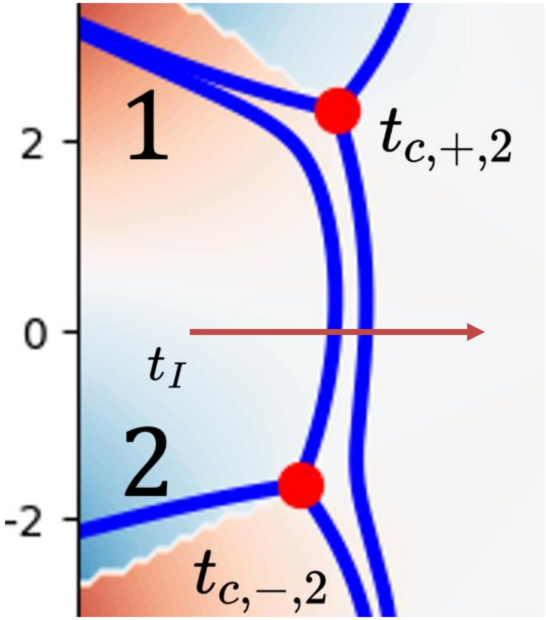



$$\begin{pmatrix} |\psi_1(t, t_I, \eta)\rangle \\ |\psi_2(t, t_I, \eta)\rangle \end{pmatrix} = \mathcal{N}(t_I, t_{c,-,2}) \begin{pmatrix} |\psi_1(t, t_{c,-,2}, \eta)\rangle \\ |\psi_2(t, t_{c,-,2}, \eta)\rangle \end{pmatrix}$$

$$\to \mathcal{N}(t_I, t_{c,-,2}) \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |\psi_1(t, t_{c,-,2}, \eta)\rangle \\ |\psi_2(t, t_{c,-,2}, \eta)\rangle \end{pmatrix}$$

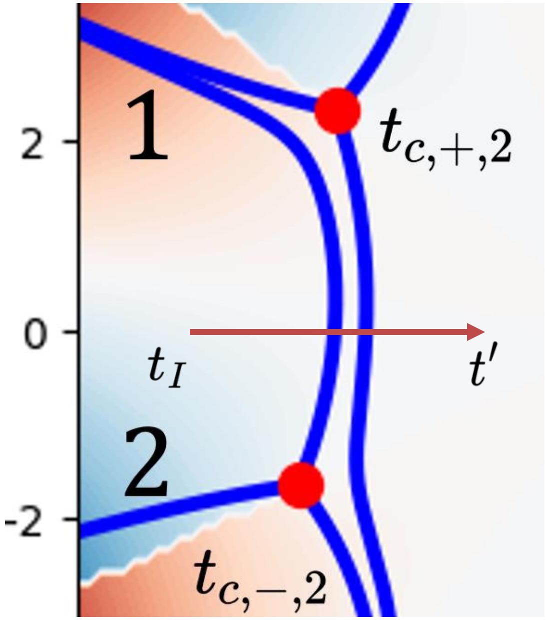
$$= \underline{\mathcal{N}(t_I, t_{c,-,2}) \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \mathcal{N}(t_{c,-,2}, t_{c,+,2})} \begin{pmatrix} |\psi_1(t, t_{c,+,2}, \eta)\rangle \\ |\psi_2(t, t_{c,+,2}, \eta)\rangle \end{pmatrix}$$

# Connection Formula for the Nonlinear LZSM Model




$$
\begin{aligned}
\begin{pmatrix} |\psi_1(t,t_I,\eta)\rangle \\ |\psi_2(t,t_I,\eta)\rangle \end{pmatrix}
&= \mathcal{N}(t_I,t_{c,-,2}) \begin{pmatrix} |\psi_1(t,t_{c,-,2},\eta)\rangle \\ |\psi_2(t,t_{c,-,2},\eta)\rangle \end{pmatrix} \\
&\to \mathcal{N}(t_I,t_{c,-,2}) \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |\psi_1(t,t_{c,-,2},\eta)\rangle \\ |\psi_2(t,t_{c,-,2},\eta)\rangle \end{pmatrix} \\
&= \mathcal{N}(t_I,t_{c,-,2}) \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \mathcal{N}(t_{c,-,2},t_{c,+,2}) \begin{pmatrix} |\psi_1(t,t_{c,+,2},\eta)\rangle \\ |\psi_2(t,t_{c,+,2},\eta)\rangle \end{pmatrix} \\
&\to \underline{\mathcal{N}(t_I,t_{c,-,2}) \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \mathcal{N}(t_{c,-,2},t_{c,+,2}) \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}} \begin{pmatrix} |\psi_1(t,t_{c,+,2},\eta)\rangle \\ |\psi_2(t,t_{c,+,2},\eta)\rangle \end{pmatrix}
\end{aligned}
$$

# Connection Formula for the Nonlinear LZSM Model




$$
\begin{aligned}
\begin{pmatrix} |\psi_1(t,t_I,\eta)\rangle \\ |\psi_2(t,t_I,\eta)\rangle \end{pmatrix}
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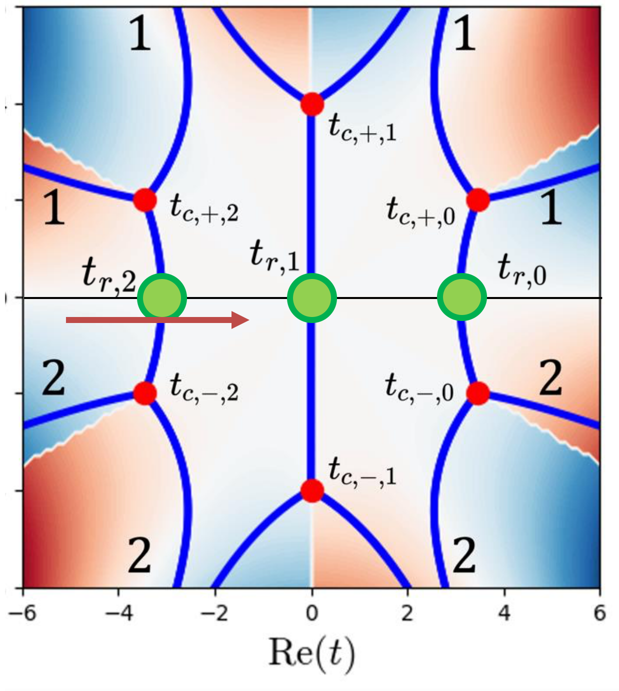
&\to \mathcal{N}(t_I,t_{c,-,2}) \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |\psi_1(t,t_{c,-,2},\eta)\rangle \\ |\psi_2(t,t_{c,-,2},\eta)\rangle \end{pmatrix} \\
&= \mathcal{N}(t_I,t_{c,-,2}) \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \mathcal{N}(t_{c,-,2},t_{c,+,2}) \begin{pmatrix} |\psi_1(t,t_{c,+,2},\eta)\rangle \\ |\psi_2(t,t_{c,+,2},\eta)\rangle \end{pmatrix} \\
&\to \mathcal{N}(t_I,t_{c,-,2}) \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \mathcal{N}(t_{c,-,2},t_{c,+,2}) \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \begin{pmatrix} |\psi_1(t,t_{c,+,2},\eta)\rangle \\ |\psi_2(t,t_{c,+,2},\eta)\rangle \end{pmatrix} \\
&= \underline{\mathcal{N}(t_I,t_{c,-,2}) \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} \mathcal{N}(t_{c,-,2},t_{c,+,2}) \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} \mathcal{N}(t_{c,+,2},t')} \begin{pmatrix} |\psi_1(t,t',\eta)\rangle \\ |\psi_2(t,t',\eta)\rangle \end{pmatrix}
\end{aligned}
$$

# Transformation to Unitary Time Evolution

$$\underline{\mathcal{N}(t_I, t_{c,-,2})}\begin{pmatrix}1 & -i\\ 0 & 1\end{pmatrix}\mathcal{N}(t_{c,-,2}, t_{c,+,2})\begin{pmatrix}1 & 0\\ i & 1\end{pmatrix}\underline{\mathcal{N}(t_{c,+,2}, t')}$$

$$= \mathcal{N}(t_I, t_{r,2})\mathcal{N}(t_{r,2}, t_{c,-,2}) \qquad = \mathcal{N}(t_{c,+,2}, t_{r,2})\mathcal{N}(t_{r,2}, t')$$




$$\mathcal{N}(a,b) = \begin{pmatrix}R_1(a,b)Q_1(a,b) & 0\\ 0 & R_2(a,b)Q_2(a,b)\end{pmatrix}, \quad R_j(a,b) = \exp\left((-1)^{j+1}\int_a^b i\eta E(s)ds\right), \quad Q_j(a,b) = \exp\left((-1)^j\int_a^b \frac{ns^{n-1}}{2E(s)}ds\right)$$

# Transformation to Unitary Time Evolution

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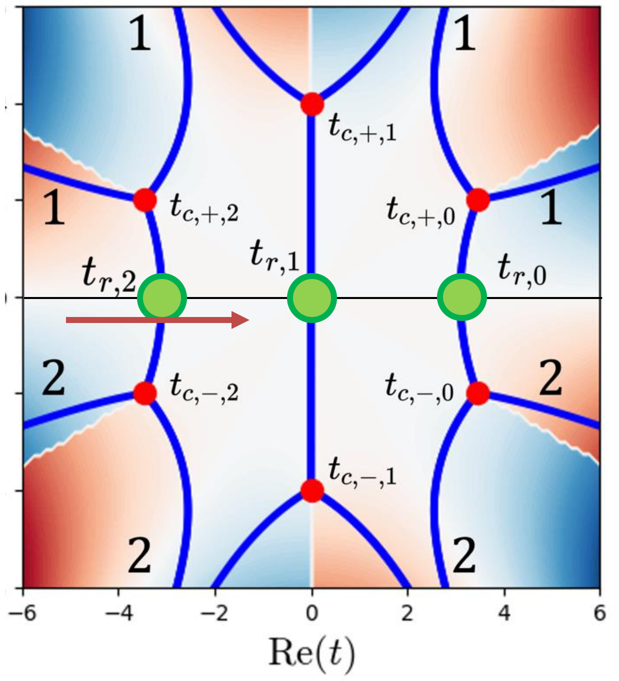
$$= \mathcal{N}(t_I, t_{r,2}) \mathcal{N}(t_{r,2}, t_{c,-,2}) \qquad = \mathcal{N}(t_{c,+,2}, t_{r,2}) \mathcal{N}(t_{r,2}, t')$$

$$\simeq \mathcal{N}(t_I, t_{r,2}) \times \begin{pmatrix} 1 & -iR_1^2(t_{r,2}, t_{c,-,2}) Q_1^2(t_{r,2}, t_{c,-,2}) \\ iR_1^2(t_{c,+,2}, t_{r,2}) Q_1^2(t_{c,+,2}, t_{r,2}) & 1 \end{pmatrix} \times \mathcal{N}(t_{r,2}, t')$$




$$\mathcal{N}(a,b) = \begin{pmatrix} R_1(a,b)Q_1(a,b) & 0 \\ 0 & R_2(a,b)Q_2(a,b) \end{pmatrix}, \quad R_j(a,b) = \exp\left((-1)^{j+1} \int_a^b i\eta E(s) ds\right), \quad Q_j(a,b) = \exp\left((-1)^j \int_a^b \frac{n s^{n-1}}{2E(s)} ds\right)$$

# Transformation to Unitary Time Evolution

$$\underline{\mathcal{N}\left(t_I, t_{r,2}\right)}\begin{pmatrix} 1 & -iR_1^2\left(t_{r,2}, t_{c,-,2}\right)r_1^2\left(t_{r,2}, t_{c,-,2}\right) \\ iR_1^2\left(t_{c,+,2}, t_{r,2}\right)r_1^2\left(t_{c,+,2}, t_{r,2}\right) & 1 \end{pmatrix}\mathcal{N}\left(t_{r,2}, t'\right)$$

$$=\underline{\mathcal{R}\left(t_I, t_{r,2}\right)\mathcal{Q}(t_I, 0)\mathcal{Q}\left(0, t_{r,2}\right)}\begin{pmatrix} 1 & -iR_1^2\left(t_{r,2}, t_{c,-,2}\right)Q_1^2\left(t_{r,2}, t_{c,-,2}\right) \\ iR_1^2\left(t_{c,+,2}, t_{r,2}\right)Q_1^2\left(t_{c,+,2}, t_{r,2}\right) & 1 \end{pmatrix}\mathcal{R}\left(t_{r,2}, t'\right)\mathcal{Q}\left(t_{r,2}, t'\right)$$

$$\mathcal{N}(a,b) = \begin{pmatrix} R_1(a,b)Q_1(a,b) & 0 \\ 0 & R_2(a,b)Q_2(a,b) \end{pmatrix}, \quad \mathcal{R}(a,b) = \begin{pmatrix} R_1(a,b) & 0 \\ 0 & R_2(a,b) \end{pmatrix}, \quad \mathcal{Q}(a,b) = \begin{pmatrix} Q_1(a,b) & 0 \\ 0 & Q_2(a,b) \end{pmatrix}$$

$$R_j(a,b) = \exp\left((-1)^{j+1}\int_a^b i\eta E(s)ds\right), \quad Q_j(a,b) = \exp\left((-1)^j\int_a^b \frac{ns^{n-1}}{2E(s)}ds\right)$$

# Transformation to Unitary Time Evolution

$$\mathcal{N}\left(t_I, t_{r,2}\right)\begin{pmatrix} 1 & -iR_1^2\left(t_{r,2}, t_{c,-,2}\right)r_1^2\left(t_{r,2}, t_{c,-,2}\right) \\ iR_1^2\left(t_{c,+,2}, t_{r,2}\right)r_1^2\left(t_{c,+,2}, t_{r,2}\right) & 1 \end{pmatrix}\mathcal{N}\left(t_{r,2}, t'\right)$$

$$=\mathcal{R}\left(t_I, t_{r,2}\right)\mathcal{Q}(t_I, 0)\underline{\mathcal{Q}\left(0, t_{r,2}\right)}\underline{\begin{pmatrix} 1 & -iR_1^2\left(t_{r,2}, t_{c,-,2}\right)Q_1^2\left(t_{r,2}, t_{c,-,2}\right) \\ iR_1^2\left(t_{c,+,2}, t_{r,2}\right)Q_1^2\left(t_{c,+,2}, t_{r,2}\right) & 1 \end{pmatrix}}\mathcal{R}\left(t_{r,2}, t'\right)\mathcal{Q}\left(t_{r,2}, t'\right)$$

$$=\mathcal{R}\left(t_I, t_{r,2}\right)\mathcal{Q}(t_I, 0)\underline{\begin{pmatrix} 1 & -iR_1^2\left(t_{r,2}, t_{c,-,2}\right)Q_1^2\left(0, t_{c,-,2}\right) \\ iR_1^2\left(t_{c,+,2}, t_{r,2}\right)Q_1^2\left(t_{c,+,2}, 0\right) & 1 \end{pmatrix}}\mathcal{R}\left(t_{r,2}, t'\right)\underline{\mathcal{Q}\left(0, t_{r,2}\right)}\mathcal{Q}\left(t_{r,2}, t'\right)$$

$$\mathcal{N}(a,b) = \begin{pmatrix} R_1(a,b)Q_1(a,b) & 0 \\ 0 & R_2(a,b)Q_2(a,b) \end{pmatrix}, \quad \mathcal{R}(a,b) = \begin{pmatrix} R_1(a,b) & 0 \\ 0 & R_2(a,b) \end{pmatrix}, \quad \mathcal{Q}(a,b) = \begin{pmatrix} Q_1(a,b) & 0 \\ 0 & Q_2(a,b) \end{pmatrix}$$

# Transformation to Unitary Time Evolution

$$\mathcal{N}(t_I, t_{r,2}) \begin{pmatrix} 1 & -iR_1^2(t_{r,2}, t_{c,-,2}) r_1^2(t_{r,2}, t_{c,-,2}) \\ iR_1^2(t_{c,+,2}, t_{r,2}) r_1^2(t_{c,+,2}, t_{r,2}) & 1 \end{pmatrix} \mathcal{N}(t_{r,2}, t')$$

$$= \mathcal{R}(t_I, t_{r,2}) \mathcal{Q}(t_I, 0) \underline{\mathcal{Q}(0, t_{r,2})} \underline{\begin{pmatrix} 1 & -iR_1^2(t_{r,2}, t_{c,-,2}) Q_1^2(t_{r,2}, t_{c,-,2}) \\ iR_1^2(t_{c,+,2}, t_{r,2}) Q_1^2(t_{c,+,2}, t_{r,2}) & 1 \end{pmatrix}} \mathcal{R}(t_{r,2}, t') \mathcal{Q}(t_{r,2}, t')$$

$$= \mathcal{R}(t_I, t_{r,2}) \mathcal{Q}(t_I, 0) \underbrace{\begin{pmatrix} 1 & -iR_1^2(t_{r,2}, t_{c,-,2}) Q_1^2(0, t_{c,-,2}) \\ iR_1^2(t_{c,+,2}, t_{r,2}) Q_1^2(t_{c,+,2}, 0) & 1 \end{pmatrix}}_{=: \mathcal{T}(t_2)} \mathcal{R}(t_{r,2}, t') \underline{\mathcal{Q}(0, t_{r,2})} \mathcal{Q}(t_{r,2}, t')$$

$$= \mathcal{R}(t_I, t_{r,2}) \mathcal{Q}(t_I, 0) \mathcal{T}(t_2) \mathcal{R}(t_{r,2}, t') \mathcal{Q}(0, t')$$

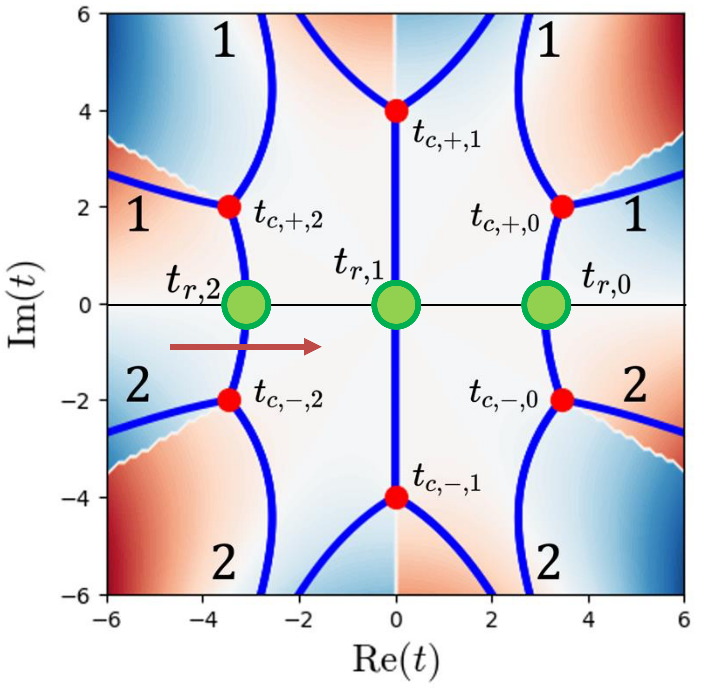
$$\mathcal{N}(a,b) = \begin{pmatrix} R_1(a,b)Q_1(a,b) & 0 \\ 0 & R_2(a,b)Q_2(a,b) \end{pmatrix}, \quad \mathcal{R}(a,b) = \begin{pmatrix} R_1(a,b) & 0 \\ 0 & R_2(a,b) \end{pmatrix}, \quad \mathcal{Q}(a,b) = \begin{pmatrix} Q_1(a,b) & 0 \\ 0 & Q_2(a,b) \end{pmatrix}$$

# Transformation to Unitary Time Evolution

$$\begin{pmatrix} |\psi_1(t, t_I, \eta)\rangle \\ |\psi_2(t, t_I, \eta)\rangle \end{pmatrix}$$

$$\to \mathcal{Q}(t_I, 0)\mathcal{R}(t_I, t_{r,2})\mathcal{T}(t_2)\mathcal{R}(t_{r,2}, t')\mathcal{Q}(0, t') \begin{pmatrix} |\psi_1(t, t', \eta)\rangle \\ |\psi_2(t, t', \eta)\rangle \end{pmatrix}$$

Adiabatic　　Adiabatic



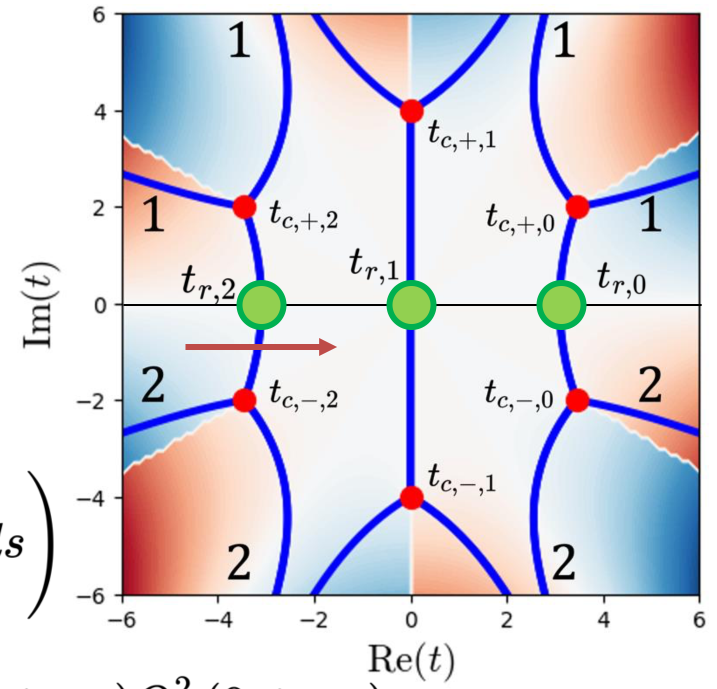

$$\mathcal{R}(a, b) = \begin{pmatrix} R_1(a, b) & 0 \\ 0 & R_2(a, b) \end{pmatrix}, \quad R_j(a, b) = \exp\left((-1)^{j+1} \int_a^b i\eta E(s) ds\right)$$

# Transformation to Unitary Time Evolution

$$\begin{pmatrix} |\psi_1(t,t_I,\eta)\rangle \\ |\psi_2(t,t_I,\eta)\rangle \end{pmatrix}$$

$$\to \mathcal{Q}(t_I,0)\mathcal{R}(t_I,t_{r,2})\mathcal{T}(t_2)\mathcal{R}(t_{r,2},t')\mathcal{Q}(0,t') \begin{pmatrix} |\psi_1(t,t',\eta)\rangle \\ |\psi_2(t,t',\eta)\rangle \end{pmatrix}$$

Adiabatic Impulse Adiabatic




$$R_1^2\,(t_{c,+,j},t_{r,j})Q_1^2\,(t_{c,+,j},0) = \exp\left(2\int_{t_{c,+j}}^{t_{r,j}} i\eta E(s)ds\right)\exp\left(-2\int_{t_{c,+j}}^{0}\frac{ns^{n-1}}{2E(s)}ds\right)$$

$$= (-1)^j i\exp\left(-2\eta\,\mathrm{Im}\int_{t_{r,j}}^{t_{c,-j}} E(s)ds\right) = R_1^2\,(t_{r,j},t_{c,-,j})Q_1^2\,(0,t_{c,-,j})$$

$$\mathcal{T}\,(t_j) = \begin{pmatrix} 1 & -iR_1^2\,(t_{r,j},t_{c,-,j})Q_1^2\,(0,t_{c,-,j}) \\ iR_1^2\,(t_{c,+,j},t_{r,j})Q_1^2\,(0,t_{r,j}) & 1 \end{pmatrix},\ R_j(a,b) = \exp\left((-1)^{j+1}\int_a^b i\eta E(s)ds\right),\ Q_j(a,b) = \exp\left((-1)^j\int_a^b \frac{ns^{n-1}}{2E(s)}ds\right)$$

# Transformation to Unitary Time Evolution

$$\begin{pmatrix}|\psi_1(t,t_I,\eta)\rangle\\|\psi_2(t,t_I,\eta)\rangle\end{pmatrix}$$

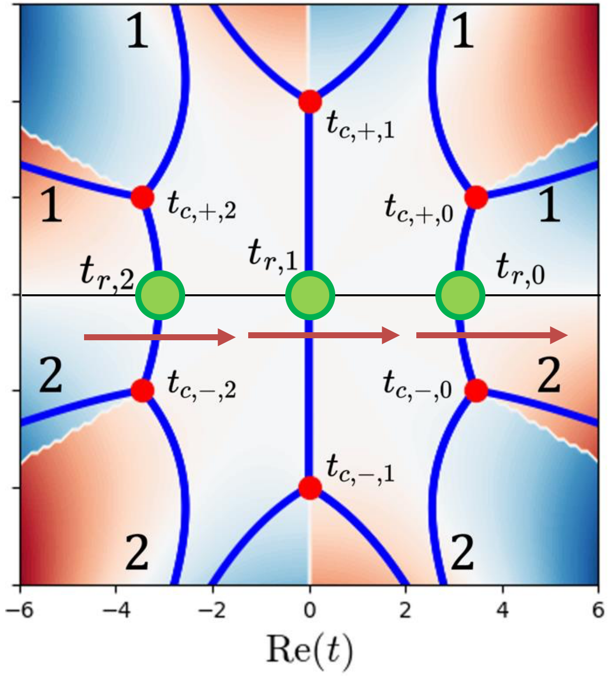
$$\to \mathcal{Q}(t_I,0)\mathcal{R}(t_I,t_{r,2})\mathcal{T}(t_2)\mathcal{R}(t_{r,2},t')\mathcal{Q}(0,t')\begin{pmatrix}|\psi_1(t,t',\eta)\rangle\\|\psi_2(t,t',\eta)\rangle\end{pmatrix}$$

Adiabatic Impulse Adiabatic

$$\to \mathcal{Q}(t_I,0)\mathcal{R}(t_I,t_{r,2})\mathcal{T}(t_2)\mathcal{R}(t_{r,2},t_{r,1})\mathcal{T}(t_1)$$

$$\times\mathcal{R}(t_{r,1},t_{r,0})\mathcal{T}(t_0)\mathcal{R}(t_{r,0},t_F)\mathcal{Q}(0,t_F)\begin{pmatrix}|\psi_1(t,t_F,\eta)\rangle\\|\psi_2(t,t_F,\eta)\rangle\end{pmatrix}$$

Adiabatic-Impulse approximation




$$\mathcal{Q}(a,b)=\begin{pmatrix}Q_1(a,b) & 0\\ 0 & Q_2(a,b)\end{pmatrix},\quad Q_j(a,b)=\exp\left((-1)^j\int_a^b\frac{ns^{n-1}}{2E(s)}ds\right)$$

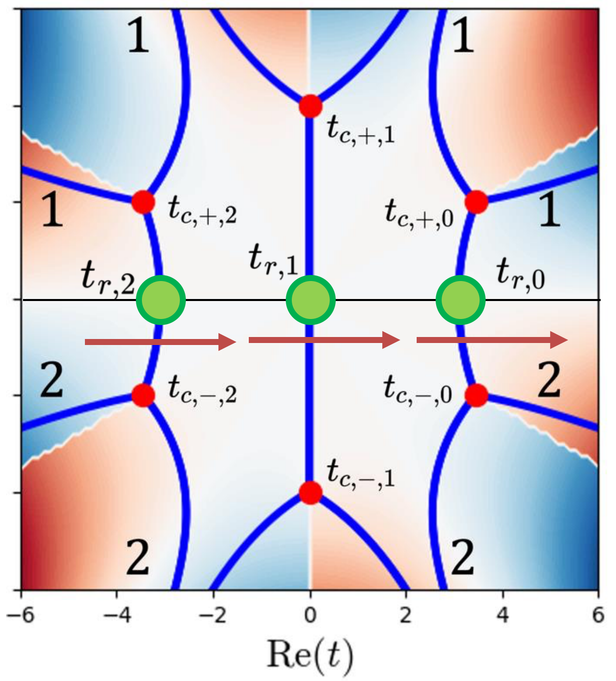
# Transformation to Unitary Time Evolution

$$\begin{pmatrix} |\psi_1(t,t_I,\eta)\rangle \\ |\psi_2(t,t_I,\eta)\rangle \end{pmatrix}$$

$$\to \underline{\mathcal{Q}(t_I,0)}\mathcal{R}(t_I,t_{r,2})\mathcal{T}(t_2)\mathcal{R}(t_{r,2},t_{r,1})\mathcal{T}(t_1)$$

$$\times \mathcal{R}(t_{r,1},t_{r,0})\mathcal{T}(t_0)\mathcal{R}(t_{r,0},t_F)\underline{\mathcal{Q}(0,t_F)}\begin{pmatrix} |\psi_1(t,t_F,\eta)\rangle \\ |\psi_2(t,t_F,\eta)\rangle \end{pmatrix}$$

$$\mathcal{Q}(0,t')\begin{pmatrix} |\psi_1(t,t',\eta)\rangle \\ |\psi_2(t,t',\eta)\rangle \end{pmatrix} = \begin{pmatrix} \exp\left(\int_{t'}^{t} i\eta E(s)\,ds\right)|E_-(t)\rangle \\ \exp\left(-\int_{t'}^{t} i\eta E(s)\,ds\right)|E_+(t)\rangle \end{pmatrix}$$




$$|\psi_j(t,t_0,\eta)\rangle = \begin{pmatrix} 1 \\ \frac{1}{\kappa}\left((-1)^j E(t) - t^n\right) \end{pmatrix} \frac{1}{\sqrt{E(t)}} \exp\left((-1)^{j+1}\int_{t_0}^{t}\left(i\eta E(s) - \frac{n s^{n-1}}{2E(s)}\right)ds\right)$$

$$\mathcal{Q}(a,b) = \begin{pmatrix} Q_1(a,b) & 0 \\ 0 & Q_2(a,b) \end{pmatrix}, \quad Q_j(a,b) = \exp\left((-1)^j \int_a^b \frac{n s^{n-1}}{2E(s)} ds\right)$$

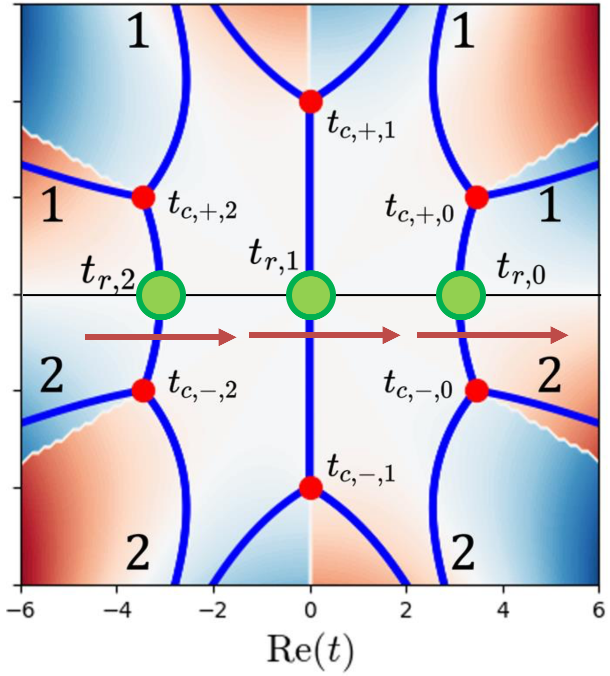
# Transformation to Unitary Time Evolution

$$\begin{pmatrix} |\psi_1(t,t_I,\eta)\rangle \\ |\psi_2(t,t_I,\eta)\rangle \end{pmatrix}$$

$$\to \mathcal{Q}(t_I,0)\mathcal{R}\,(t_I,t_{r,2})\mathcal{T}\,(t_2)\mathcal{R}\,(t_{r,2},t_{r,1})\mathcal{T}\,(t_1)$$

$$\times\mathcal{R}\,(t_{r,1},t_{r,0})\mathcal{T}\,(t_0)\mathcal{R}\,(t_{r,0},t_F)\mathcal{Q}(0,t_F)\begin{pmatrix} |\psi_1(t,t_F,\eta)\rangle \\ |\psi_2(t,t_F,\eta)\rangle \end{pmatrix}$$

$$\mathcal{Q}(0,t')\begin{pmatrix} |\psi_1(t,t',\eta)\rangle \\ |\psi_2(t,t',\eta)\rangle \end{pmatrix} = \begin{pmatrix} \exp\left(\int_{t'}^{t} i\eta E(s)\,ds\right)|E_-(t)\rangle \\ \exp\left(-\int_{t'}^{t} i\eta E(s)\,ds\right)|E_+(t)\rangle \end{pmatrix}$$



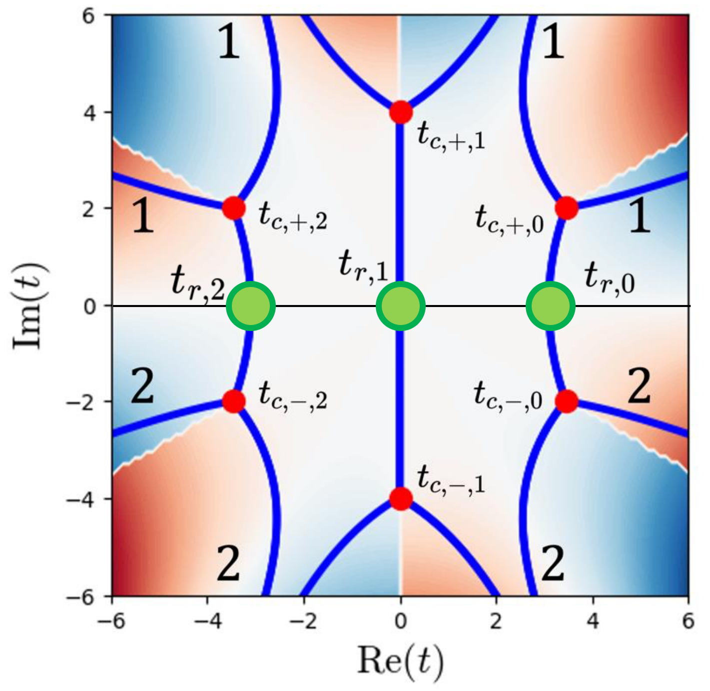

$$\begin{pmatrix} |E_-(t_I)\rangle \\ |E_+(t_I)\rangle \end{pmatrix} \to \mathcal{R}\,(t_I,t_{r,2})\mathcal{T}\,(t_2)\mathcal{R}\,(t_{r,2},t_{r,1})\mathcal{T}\,(t_1)\mathcal{R}\,(t_{r,1},t_{r,0})\mathcal{T}\,(t_0)\mathcal{R}\,(t_{r,0},t_F)\begin{pmatrix} |E_-(t_F)\rangle \\ |E_+(t_F)\rangle \end{pmatrix}$$

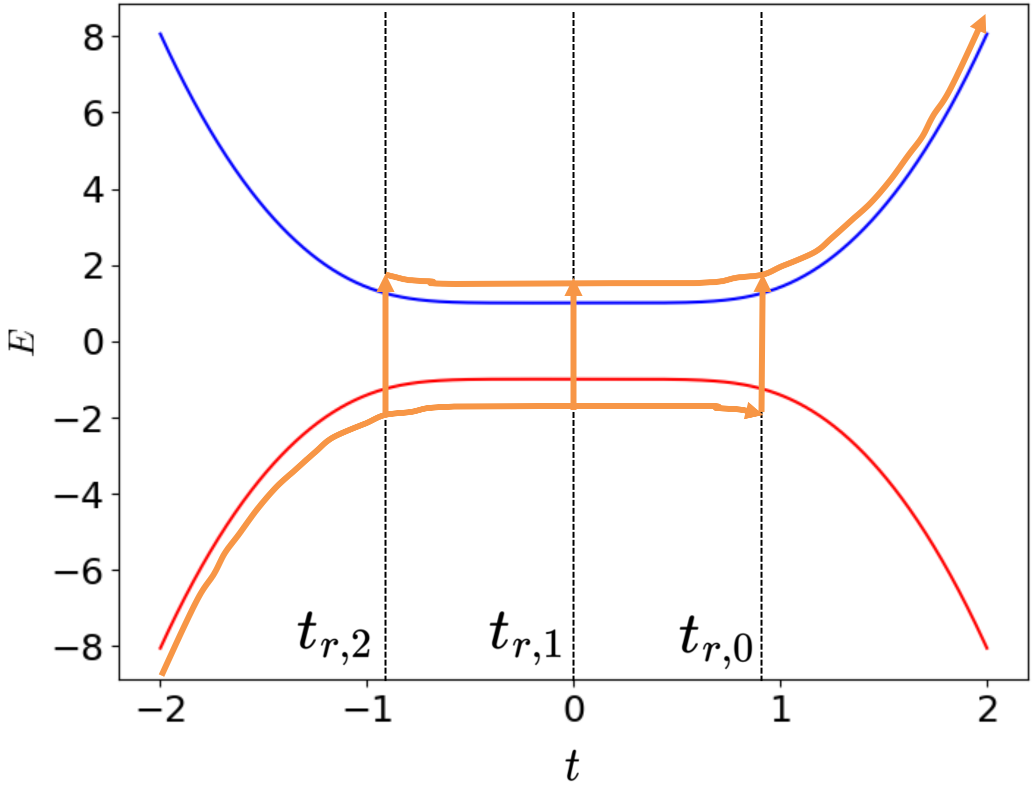
**TS**, Taniguchi, Iwamura, PRA (2024)

# Interpretation of Adiabatic-Impulse Approximation

$$|E_-(t_I)\rangle \to e^{i\eta\int_{t_I}^{t_F}E(s)ds}|E_-(t_F)\rangle + \sum_{k=0}^{n-1}(-1)^k \boxed{e^{i\eta\int_{t_I}^{t_{r,k}}E(s)ds}e^{-i\eta\int_{t_{r,k}}^{t_F}E(s)ds}}\boxed{e^{-2\eta\,\mathrm{Im}\int_{t_{r,k}}^{t_{c,+,k}}E(s)ds}}|E_+(t_F)\rangle$$

Dynamical phase　　Transition amplitude



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# "Time-dependent Resonance" Hamiltonian

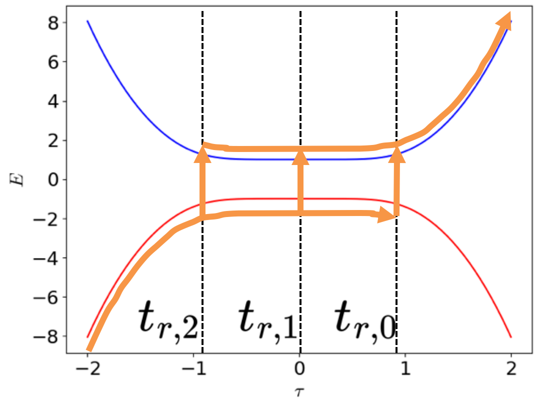
$$H(t) = \left( t^n + \sum_{k=0}^{n-1} A_k(t) \sin\phi\,(t, t_{r,k}) \right) \sigma_z + \kappa\sigma_x$$

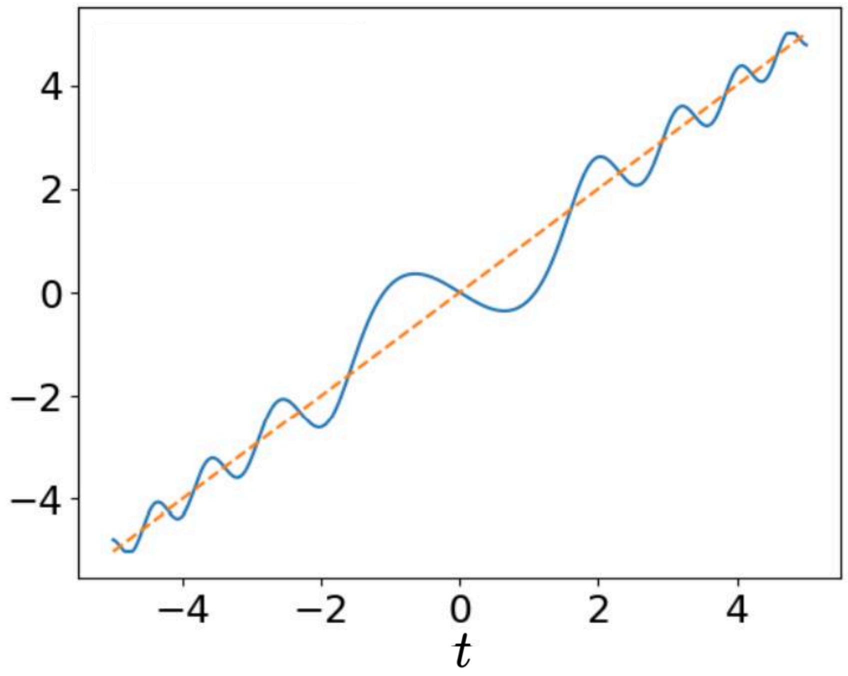
Free Hamiltonian : $\mathcal{H}(t)$

$$A_k(t) = \frac{-\alpha_k}{\mathcal{E}(t)}$$

$$\mathcal{E}(t) = \sqrt{t^{2n} + \kappa^2}$$

$$\phi\,(t, t_0) = 2\int_{t_0}^{t} \mathcal{E}(s)ds$$




**Conclusion**

In the adiabatic region,

only the free Hamiltonian → transition probability ≃ 0

Hamiltonian with an appropriate $\alpha_k$ → transition probability = 0 (≃ optimal control)

# Analysis of Dynamics

TS, PRA (2025)

$$H(t) = \underbrace{\left(t^n + \sum_{k=0}^{n-1} A_k(t) \sin\phi\left(t, t_{r,k}\right)\right)\sigma_z + \kappa\sigma_x}_{\mathcal{H}(t)}$$

**Transition amplitude** (up to 1st order)

$$P_e \simeq \left| \left\langle \mathcal{E}_+\left(t_F\right) \right| \mathcal{U}\left(t_f, t_0\right) \left| \mathcal{E}_-\left(t_0\right) \right\rangle - i \left\langle \mathcal{E}_+\left(t_F\right) \right| \mathcal{U}\left(t_F, t_I\right) \int_{t_I}^{t_F} F(s) ds \left| \mathcal{E}_-\left(t_I\right) \right\rangle \right|^2$$

0th order : Exact WKB analysis

$$F(s) = \mathcal{U}^\dagger(s, t_I)\sigma_z \mathcal{U}(s, t_I) \sum_{k=0}^{n-1} A_{n,k}(s) \sin\phi_n\left(s, t_{r,k}\right)$$

1st order

$$\simeq -\frac{\kappa}{2} e^{-i\int_{t_I}^{t_F} \mathcal{E}(s)ds} \sum_{k=0}^{n-1} e^{i\phi(t_{r,k}, t_I)} \int_{t_I}^{t_F} \frac{\alpha_k}{\mathcal{E}^2(s)} ds$$

Adiabatic approximation

$$\simeq \left| \sum_{k=0}^{n-1} e^{i\int_{t_I}^{t_{r,k}} \mathcal{E}(s)ds} e^{-i\int_{t_{r,k}}^{t_F} \mathcal{E}(s)ds} \left( (-1)^k e^{-2\,\mathrm{Im}\int_{t_{r,k}}^{t_{c,+,k}} \mathcal{E}(s)ds} - \frac{\alpha_k \kappa}{2} \int_{t_I}^{t_F} \frac{ds}{\mathcal{E}^2(s)} \right) \right|^2$$

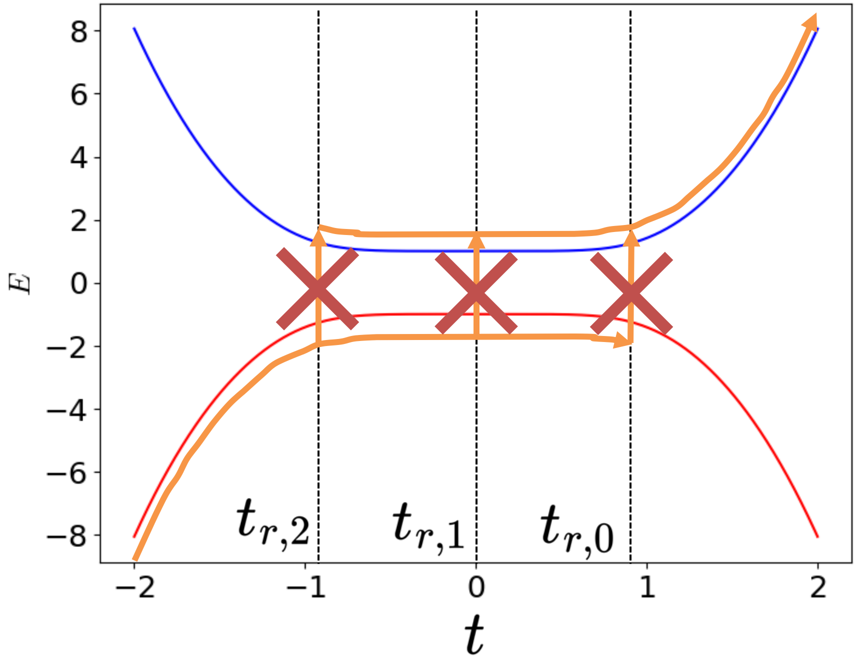
# Analysis of Dynamics

**TS**, PRA (2025)

$$H(t) = \underbrace{\Bigg(t^n + \sum_{k=0}^{n-1} A_k(t)\sin\phi\left(t, t_{r,k}\right)\Bigg)\sigma_z + \kappa\sigma_x}_{\mathcal{H}(t)}$$

**Transition amplitude** (up to 1st order)

$$P_e \simeq \left| \sum_{k=0}^{n-1} e^{i\int_{t_I}^{t_{r,k}} \mathcal{E}(s)ds} e^{-i\int_{t_{r,k}}^{t_F} \mathcal{E}(s)ds} \left( (-1)^k e^{-2\,\mathrm{Im}\int_{t_{r,k}}^{t_{c,+,k}} \mathcal{E}(s)ds} - \frac{\alpha_k \kappa}{2} \int_{t_I}^{t_F} \frac{ds}{\mathcal{E}^2(s)} \right) \right|^2$$




**By choosing an appropriate $\alpha_k$, the transition probability can be made 0.**

# Quantum Optimal Control

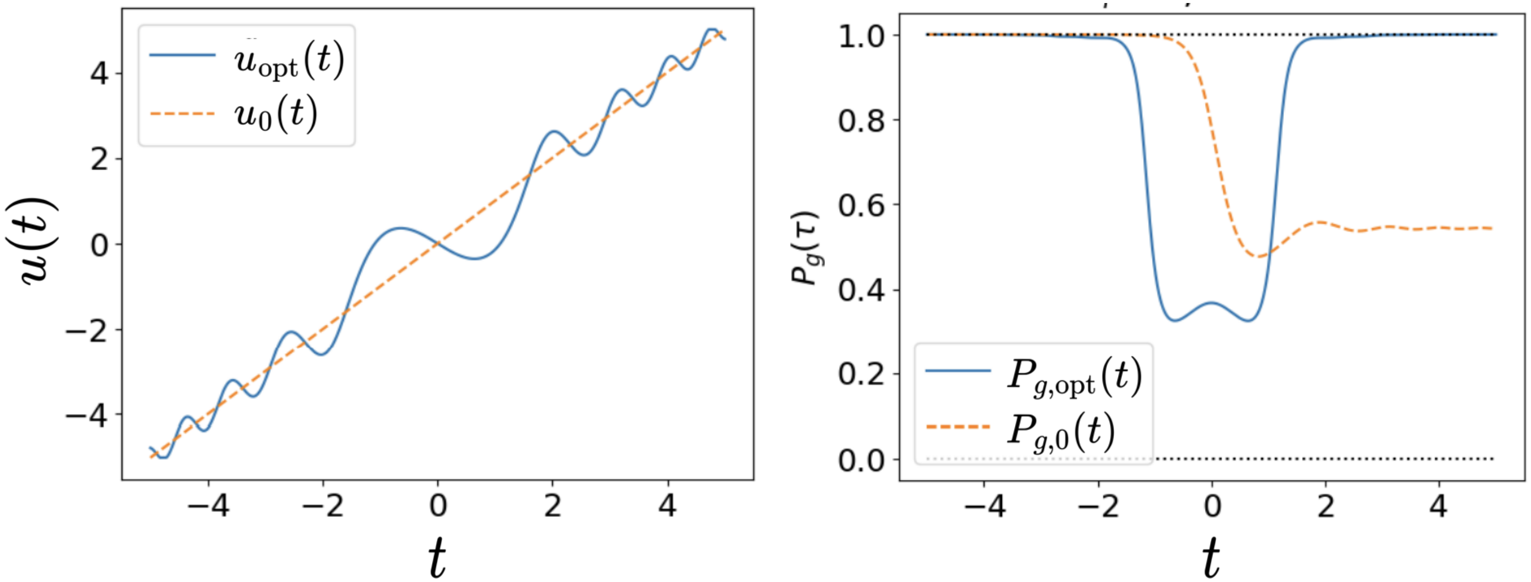
Optimal control: the variation of the functional is zero Brady, et al. PRL (2021)

$$J[|x(t)\rangle, \langle k(t)|, u(t)] = \langle x(T)|H_C|x(T)\rangle + \int_{-T}^{T} dt \left( \langle k(t)| \left[ -\frac{d}{dt} - iH(t) \right] |x(t)\rangle + \text{ c.c.} \right)$$

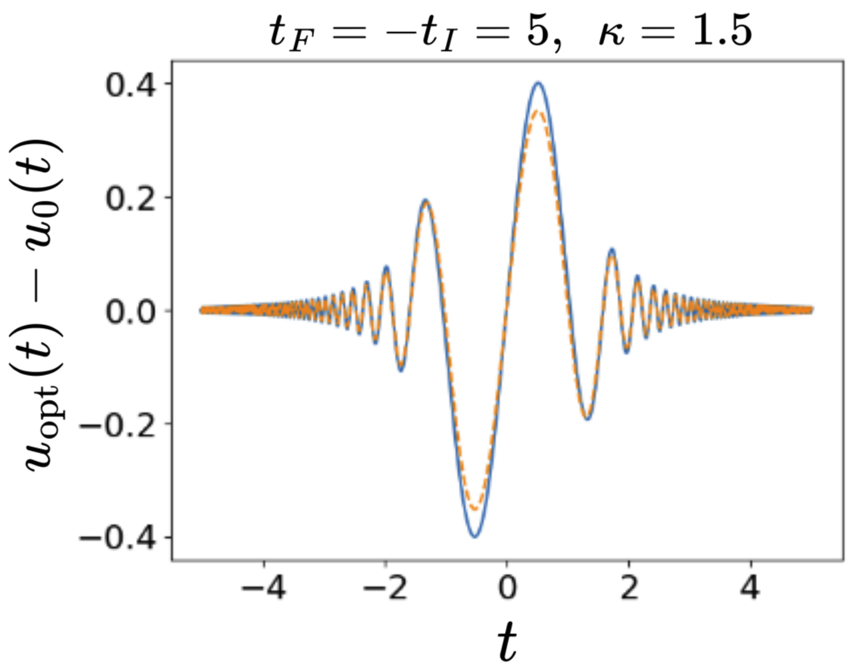
$$H(t) = u(t)\sigma_z + \kappa\sigma_x, \quad H_C = u_0\,(t_F)\sigma_z + \kappa\sigma_x$$

- Initial state: ground state

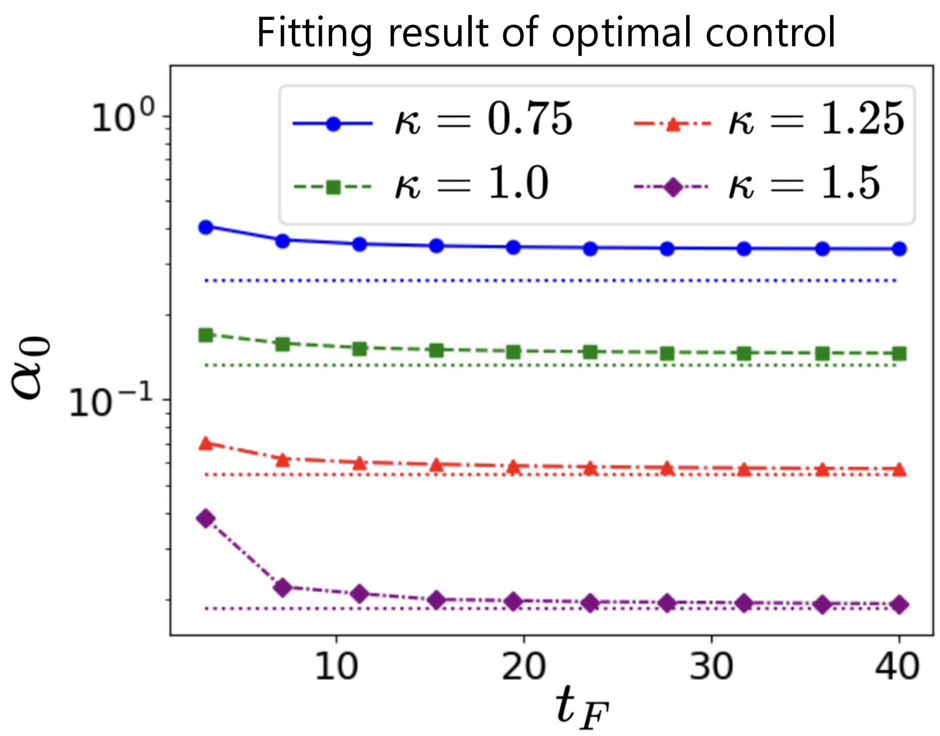
$$t_F = -t_I = 5, \;\kappa = 0.5$$



# Comparison of Optimal Control and Time-Dependent Resonance Protocol




Solid：optimal control
Dashed：time-dependent resonance




Dotted：time-dependent resonance

The two coincide in the long-time adiabatic limit